Problem B: The Longest Lasting Sandcastle(s)

SUMMARY

To create a mathematical model of the longest lasting sandcastle under the provided conditions, our team utilized geometric properties of 3-dimensional shapes to create a semiempirical model. Through extensive analysis and synthesis of research, we determined the main elements of consideration for our model, the shape of the foundation, the sliding friction between wet and dry sand, and the stress from environmental conditions. The main environmental conditions to consider include erosion due to ties and wear from weather conditions such as rain. Our analysis of the best 3-dimensional geometric shapes produced two leading models, a square pyramidal frustum and a conical frustum. We started with a basic model of a square pyramid frustum (a strong foundation on its own) and arguably as strong as the conic frustum and increased the number of vertical faces to eventually reach the model of a conical frustum. Through our research of sliding friction, we were able to understand the relationship between stability and the stickiness of wet sand, however, due to complexities, we ended with a result that could be supported by research with experimental evidence. Ultimately, we decided the best 3-dimensional geometric shape to use as a sandcastle foundation, that will last the longest period of time exposed to the elements of nature on a seashore, is a conical frustum with a 1:8 ratio of water to sand.

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INTRODUCTION

Does there exist a *best* 3-dimensional geometric shape to use for a sandcastle foundation? Regardless of one's interpretation of a sandcastle foundation, there must exist a foundation that is above the rest, serving as a timeless base for our sandy creations.

Initially, our team wanted to approach this problem using only textbook methods–looking at numerous studies, research papers, architectural diagrams, articles, and much more. However, we soon realized that we needed an approach that was much more dynamic, and thus we decided to incorporate our research with our personal creativity and historical references. Throughout multiple experiments, we derived two models and came to the conclusion that the best 3-dimensional geometric shape to use for a sandcastle foundation is a conical frustum.

Background:

Before describing our specific solution in any detail, it is important to provide some background information on our topic in order to show and describe some of what we learned while researching it.

Scientists have been approaching the perfect sandcastles through many aspects. Some researches stated that when the proportion of sand to water is 8:1, the sandcastle would be the most stable. A group of scientists from the Massachusetts Institute of Technology (MIT) experimented with the proportion and developed a mathematical model which proved that since when adding liquids to the sand, which is referred to as granular materials, would form "bridges" between grains, so that there will appear an attractive force holding the sand even tighter (16).

The studies on constructing perfect sand castles started roughly around 2008 and it has been constantly modified for the best foundation for the perfect long-lasting castles. Most studies that we found are aiming at the maximum height of the castles and the best size for the castle. According to the U.S. national center for biotechnology information, the height of the sandcastles are found to increase as ⅔ power of the base radius of column and the optimum strength for wet sand is approximately 1% of the liquid volume fraction. (https://www.ncbi.nlm.nih.gov/pmc/articles/PMC3412320/) In fact, there is not much quantitative research on how the shape of the sandcastle foundation would affect the stability. And the article also pointed out that

*A recently introduced model for the strength of wet granular matter*13 *assumes that when one adds a volume of liquid to grains, the capillary attractive force and elastic response from the Hertz contact between two spheres will be balanced. As two beads are always separated by at* *least the surface roughness, below a critical liquid volume fraction about 0.2%, the bridges between the beads cannot form. At higher volume fractions, the bridge force is dominated by the curvature of the meniscus and at even higher volume fractions the bridges start merge into larger pockets of fluid*[13]*. The macroscopic shear modulus G of a macroscopic cube of dimension L containing a large amount of grains can be defined as the ratio of stress and strain:*

$$G = 2(1+v)\frac{F_{strain}/L^2}{\Delta x/L},$$

where Δx/L is the strain, Fstrain/L2 the stress and v ≈ 0.5 the Poisson ratio.

How does wet sand stick together?

This relationship is significant because it can help us understand the ideal water-sand ratio. When sand is dry, there are large gaps between the small granule particles that cause extreme slipping between one another. The addition of water to sand bridges these gaps and allows for greater interlocking between granule particles. These bridges are referred to as "interstitial liquid bridges" and can be explained by the surface tension of water.[13]

To understand mathematically how adding water affects the stickiness of sand, we can look to the length between two sand grains when dry and then when wet. If we use the Hertz-Mindlin's Formulation to understand the stiffness of interaction between sand grains, shear stiffness (the strain of parallel internal surfaces moving past one another) and normal stiffness (the force-displacement of two rough surfaces in contact with one another) we can see that. [15]

Normal stiffness: $k_n = k_{n0}(f/Gl^2)^n$

Shear Stiffness: $k_r = k_{r0}\,(f/Gl^2)^n$

Where f is the contact force between grains, G is the elastic modulus of the grains, l is the length between two particles, and n is a material constant

The significance of viewing the Hertz-Mindlin's Formulation is that for dry sand, we can see that the length between the two grains (if we assume they are the same size) from center to center will be the diameter of one grain. Whereas, when we wet the sand, the length between the grains will be less than the diameter of one grain. This is because the addition of water increases the stiffness of sand, meaning the friction between each particle increases. The smallest distance between grains would result in the greatest stiffness (as long as all other variables remained constant). To find the ideal mixture of sand and water for our foundation, we could find the smallest distance between sand grains to maximize the stiffness and utilize this distance to find the maximum force between sand grains when the surface tension of water is present. This can be related to the electrostatic force between sand grains stuck together due to the polarity of water molecules (where the force is inversely related to the distance of separation between objects). This would give us the water to sand ratio for two grains which could then be macroscopically evaluated to fit our sandcastle foundation.

When researching sand, we must consider the properties of the dry material without the mixture of water first. A common sand structure found in nature is a sand dune. For a sand dune to remain intact (and not flatten to the relative horizontal plane), it can be understood that the dune relies on internal friction for support. This internal friction concerns the granule material that comprises the dune, which can be directly correlated to the size and surface of the material. This is inferred from our understanding of static friction of materials which relies on the force between two objects that prevents slippage and is correlated to the coefficient of static friction of specific materials. When considering sand dunes, we can understand that as the static friction between grains increases, the slope would increase and vice versa. This allows us to intuitively understand the slope of a dune which can also be referred to as the angle of repose. This angle tells us the “steepest angle of descent relative to the horizontal plane to which a material can be piled without slumping” (12). The significance of understanding the angle of repose allows us to analyze the relationship between the base of a dune and the height of the dune, in which we can use the Pythagorean theorem to find. It is important to note that the angle of repose is related to “the density, surface area, shapes of particles, coefficient of friction of the material, and gravity” (12). The angle of repose can be found by

$$\theta = arctan(\frac{h}{r})$$

where we define h as the height of the conic structure, and r as the radius of the circular base

We are also able to understand the relationship between the angle of repose and the friction between sand grains by using the state of packing void ratio (volume of a section divided by the total volume) and the critical void ratio (volume does not change, there is constant distortion).[15] This tells us that the angle of repose is dependent on the friction between particles, or how well they interlock with one another.

$$tan(\theta) = (c/e)^m tan(\gamma)$$

where θ is the angle of repose, ec is the critical void ratio, e is the packing void ratio, m is the material constant (of sand in this case), and γ is the interparticle friction angle which is constant for a defined material

Intuitively when we stand on a sand castle, or apply force to it, the foundation (or packing structure) falls apart. This means that both the angle of repose and interparticle friction angle will decrease resulting in strain softening meaning that the contact between particles decreases.[17]

The smaller the grain is (<0.6mm), the more significant its surface energy forces become as the grains move against one another frequently. There does not exist a complete model for the microstructure of assembled sand grains because the deformation behavior of sand cannot be found by using strictly particle properties of the sand. The rotation of particles such as sand, potentially introduces micropolar strain and stress which we are unable to evaluate; leading us to make the assumption to neglect particle rotation. Because of this, it is apparent that we should approach this problem with a semiempirical model in which we can assume idealized conditions.[15]

3-Dimensional Shapes

When analyzing shapes, the strongest structures (in nature) are the sphere, the triangle: as a pyramid with a square and triangle base (tetrahedron), the catenary curve, and the hexagon. The circle (and sphere) distribute stress uniformly throughout the arc rather than at any specific point. The triangle (and pyramids) has one unique form for a triangle of three rigid sides and fixed angles, that cannot be modified without breaking a joint. A catenary curve is a natural form in nature usually occurring by wire freely hanging from two endpoints. It is the “strongest shape which supports only its own shape”. Hexagons are the most efficient shape, as a grid of hexagons utilizes the shortest possible line to fulfill the largest area with the fewest number of hexagons. Because of this compressible nature, hexagons are able to support a large amount of weight and pack with minimal gaps between shapes. [11]

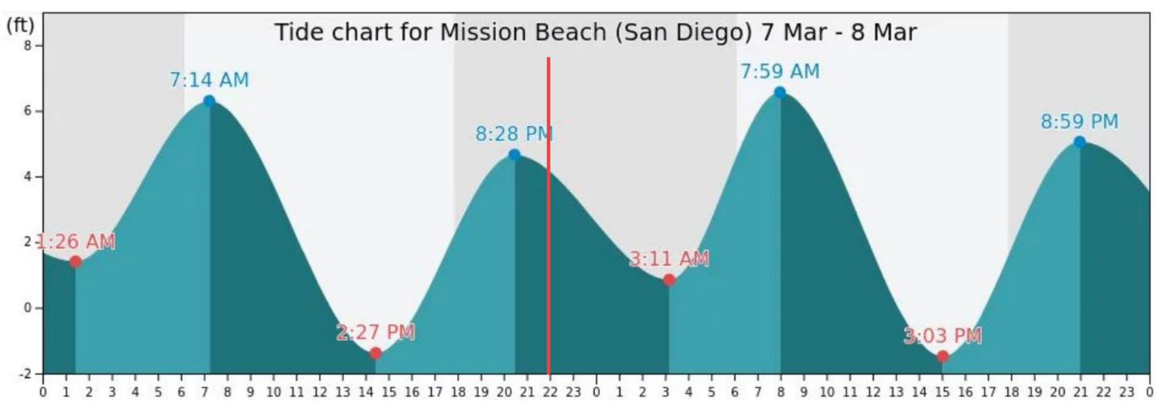
When considering 3-dimensional shapes constructed with granule material, sand, we must consider how these shapes will uphold.

The purpose of doing such research is meaningful. Peter Schiffer from Pennsylvania State University, wrote: "The work is an important advance and opens the door to more systematic studies of how the addition of interstitial liquid affects the dynamics of such materials."

From all data and resources we have, we determined the factors that we should work on in order to find the perfect shape of the foundation. We decided to start testing.

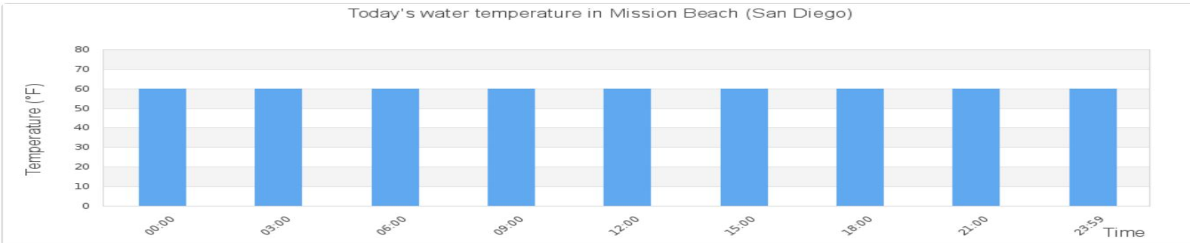
Assumptions:

1. The size of the foundation is variable, scaled to the size of the sandcastle.
2. The time for all tests is March 7th at 8:28PM.
3. All sandcastles are built at roughly the same distance from the water on the same beach.
4. All sandcastles are built with the same type of sand, roughly the same amount of sand, same water-to-sand proportion.
5. The density for the sandcastle would be the same.
6. There are no other additives or materials in the foundation of the sand castle.
7. The air humidity is fixed at 76%
8. The wind is fixed at 2mph.
9. The average water temperature is fixed at 60F. (https://www.tideschart.com/United-States/California/San-Diego-County/Mission-Beach-(San-Diego)/)
10. Tides are fixed at 4.66ft and waves would be constantly the same for all tests. (https://www.tideschart.com/United-States/California/San-Diego-County/Mission-Beach-(San-Diego)/)
11. The granules of sand are the same size and spherical shape



(google images for local tide charts, foreseeable forecasts and water temperature)

Today's water temperature in Mission Beach (San Diego)



Saturday 7th of March 2020, 9:57 pm. The sun rose at 6:10 am and the sun went down at 5:51 pm. Today there was 11 hours and 41 minutes of sun and the average temperature is 60°F. At the moment the current water temperature is 60°F. and the average water temperature is 60°F.

Mission Beach (San Diego) tide table for the next week

DAY	TIDES FOR MISSION BEACH (SAN DIEGO)					
	1st TIDE	2nd TIDE	3rd TIDE	4th TIDE		
7 Sat	01:26 h v 1.41 ft	07:14 h ^ 6.3 ft	14:27 h v -1.38 ft	20:28 h ^ 4.66 ft	^ 06:10 h	v 17:51 h
8 Sun	02:11 h v 0.85 ft	07:59 h ^ 6.56 ft	15:03 h v -1.48 ft	20:59 h ^ 5.05 ft	^ 06:08 h	v 17:52 h
9 Mon	02:55 h v 0.36 ft	08:44 h ^ 6.56 ft	15:38 h v -1.31 ft	21:32 h ^ 5.38 ft	^ 06:07 h	v 17:52 h
10 Tue	03:41 h v -0.03 ft	09:30 h ^ 6.23 ft	16:13 h v -0.95 ft	22:07 h ^ 5.61 ft	^ 07:06 h	v 18:53 h
11 Wed	04:29 h v -0.23 ft	10:18 h ^ 5.68 ft	16:49 h v -0.36 ft	22:44 h ^ 5.71 ft	^ 07:05 h	v 18:54 h
12 Thu	05:21 h v -0.23 ft	11:09 h ^ 4.92 ft	17:25 h v 0.33 ft		^ 07:04 h	v 18:54 h

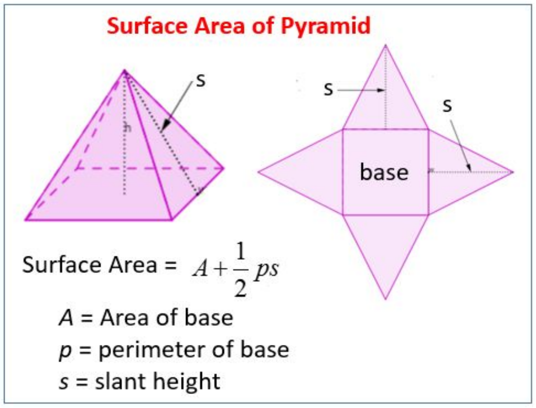
MODEL(S)

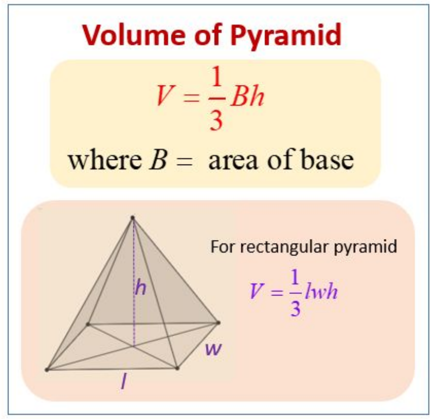
Predictions:

1. The best 3-dimensional geometric shape to use as a sandcastle foundation would be the conical frustum. The slanted faces of these shapes would allow for the least amount of contact with water or wind, reducing the overall wear on the shape.
2. The optimal sand-to-water mixture would be 1 part water to 8 parts sand (well-mixed), with a high enough density to not wear over time.
3. The optimal density of the foundation would be extremely high.
4. When it rains a certain amount, it will make the sandcastle more stable, as the shape will not dry out on the surface, which will lead to further erosion.

Preliminary model research:

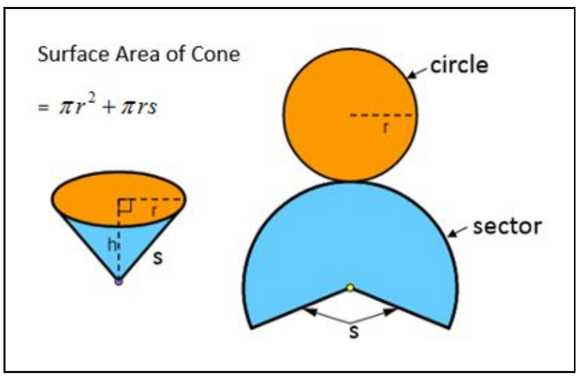
A right square pyramid has a surface area and volume:

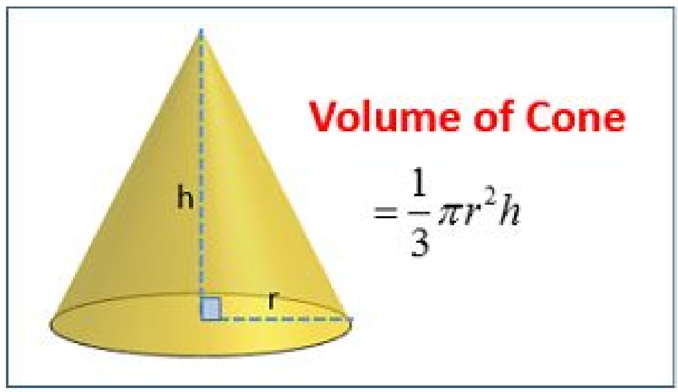




(https://www.onlinemathlearning.com/surface-area-pyramid.html & https://www.onlinemathlearning.com/volumes-in-geometry.html)

A right circular cone has a surface area and volume:



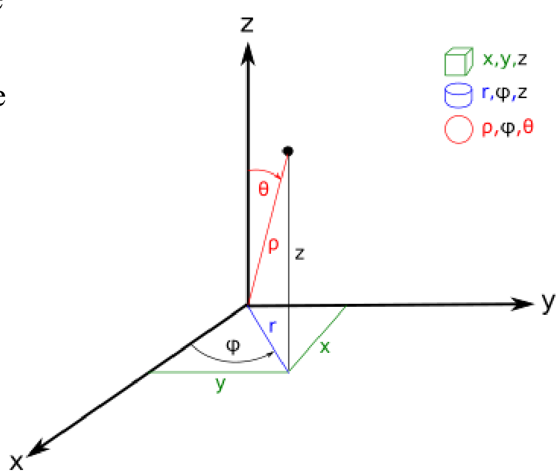


(https://www.onlinemathlearning.com/surface-area-cone.html & https://www.onlinemathlearning.com/volume-cone.html)

Note, since the size (volume and surface area) of the castle is dependent on the size of the foundation, we will assume that the size of the foundation is variable–scaled to the size of the sandcastle. The size of the sandcastle affects the strength of the foundation over time, especially under varying conditions.

Defining Coordinate Systems

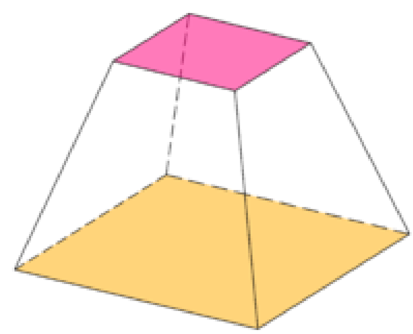
The coordinate systems utilized within our models and model analysis include: (https://planetcalc.com/users/1/1540029221.png)



In which cartesian and cylindrical coordinate systems are utilized for the analysis of our models.

Model 1: Pyramidal Frustum, the Square Pyramid

For a right pyramidal frustum, let s be the slant height, h be the height, p_1 the bottom base perimeter, p_2 be the top base perimeter, A_1 be the bottom area, A_2 be the top area. The surface area of the sides and volume of a pyramid frustum are given by:



$$S = \frac{1}{2}(p_1 + p_2)s$$

$$V = \frac{1}{3}h(A_1 + A_2 + \sqrt{A_1 A_2})$$

Further, the side surface area of a right square pyramidal frustum (right) is:

$S_4 = 2(a+b)\sqrt{\frac{1}{2}(a-b)^2 + h^2}$, where h is the height and a and b are the side lengths of the polygons (at the bases of the frustum).

The volume of a right square pyramidal frustum is:

$$V_4 = \frac{1}{3}(a^2 + ab + b^2)h$$

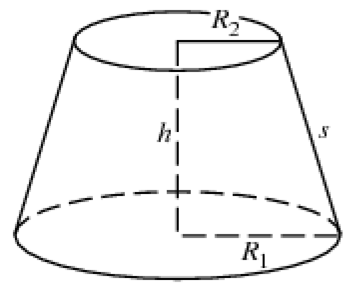
The 4 in S_4 and V_4 corresponds to the number of faces on the frustum, creating a square frustum. (https://mathworld.wolfram.com/PyramidalFrustum.html)

<u>Going from a Square Frustum to a Conical Frustum</u>

In our model, in order to go from a square frustum to a conical frustum, we used this method:

Firstly, understand the concept of the construction of the constant number π and secondly, be able to transform the area of a square to the area of a circle. As the number of sides of a polygon approaches infinity, the internal angle between sides approaches 180 degrees, and thus the shape of the polygon approaches that of a circle. Thus, if we take the square frustum and increase the number of sides of each polygon across its height, we would obtain a conical frustum. Essentially, we are increasing the number of sides of the square frustum to obtain a conical frustum.

Model 2: Conical Frustum, the Right Circular Conical Frustum



A conical frustum (right) is a frustum created by slicing the top off a cone (with the cut made parallel to the base). For a right circular cone, let s be the slant height and R_1 and R_2 the base and top radii respectively.

Then: $s = \sqrt{(R_1 - R_2)^2 + h^2}$ and the surface area (A), not including the top and bottom circles, is:

$A = \pi(R_1 + R_2)s = \pi(R_1 + R_2)\sqrt{(R_1 - R_2)^2 + h^2}$.

The volume of a conical frustum is: $V = \pi\int_0^h [r(z)]^2 dz$, where $r(z) = R_1 + (R_2 - R_1)\frac{z}{h}$.

So, $V = \pi\int_0^h [r(z)]^2 dz = \pi\int_0^h [R_1 + (R_2 - R_1)\frac{z}{h}]^2 dz = \frac{1}{3}\pi h(R_1^2 + R_1R_2 + R_2^2)$.

(https://mathworld.wolfram.com/ConicalFrustum.html)

SOLUTION(S)

This is where we explain why one of the models we described is best for the foundation of a sandcastle. Some of the reasons for its appropriateness would be related to the points made/described in the introduction, background, and our assumptions/research.

Our solution on optimum sand-to-water of 8:1 takes advantage of the following two unoriginal scientific analysis:

1. https://www.telegraph.co.uk/news/worldnews/northamerica/usa/1499523/Scientists-find-formula-for-perfect-sandcastle.html
2. https://www.pbs.org/newshour/science/build-perfect-sandcastle-science

To understand mathematically how adding water affects the stickiness of sand, we can look at the length between two sand grains when dry and then when wet. If we use the Hertz-Mindlin's Formulation to understand the stiffness of interaction between sand grains, shear stiffness (the strain of parallel internal surfaces moving past one another) and normal stiffness (the force-displacement of two rough surfaces in contact with one another) we can see that. [15]

Normal stiffness: $k_n = k_{n0}(f/Gl^2)^n$

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Where f is the contact force between grains, G is the elastic modulus of the grains, l is the length between two particles, and n is a material constant

The significance of viewing the Hertz-Mindlin's Formulation is that for dry sand, we can see that the length between the two grains (if we assume they are the same size) from center to center will be the diameter of one grain. Whereas, when we wet the sand, the length between the grains will be less than the diameter of one grain. This is because the addition of water increases the stiffness of sand, meaning the friction between each particle increases. The smallest distance between grains would result in the greatest stiffness (as long as all other variables remained constant).

To find the ideal mixture of sand and water for our foundation, we could find the smallest distance between sand grains to maximize the stiffness and utilize this distance to find the maximum force between sand grains when the surface tension of water is present. This can be related to the electrostatic force between sand grains stuck together due to the polarity of water molecules (where the force is inversely related to the distance of separation between objects). This would give us the water to sand ratio for two grains which could then be macroscopically evaluated to fit our sandcastle foundation.

However, because this process is highly unrealistic for the provided time limits, we have evaluated numerous resources that have suggested the optimal ratio of water to sand for stickiness is about 1 part water to 8 parts sand.

Comparison of solutions:

In both of our proposed models, we have chosen 3-dimensional shapes that include two bases of varying size. In the first model of a pyramidal frustum with square bases, this creates slanted sides that connect to form lines. In the second model of the conical frustum, the cross-sectional areas vary, but the shapes (along the cross-sections) remain a circle along a constant axis. The benefit of choosing these slanted angles allows us to consider the force of gravity along the foundation. If the bases were directly stacked on one another, we would consider any point on the top surface to experience the same force of gravity (as the shape is uniform along the z-axis). Utilizing slanted surfaces allows us to distribute the amount of sand at each point along the volume so that the height varies with position. When considering our 3-dimensional shapes, we had to account for environmental factors that could affect our structures such as erosion or rain. By choosing slanted sides, we have ensured that if one portion of one of the sides is affected by rain or tides, the entire structure will not collapse. If our sides were aligned directly beneath the bases, a tide coming into contact with the side of the structure would damage the side and the sand directly above it is subject to falling. With the use of slanted sides and varying cross-sectional areas, if a wave were to skim the base, the structure would remain intact, and the risk of collapsing is much smaller because the top base is still supported by a wider base. By having this larger base, the center of mass of the foundation is closer to the horizontal plane on which it rests. The distribution of mass is skewed this way to prevent toppling when the castle is added atop the foundation. We can find the centroid of a 3-dimensional solid by finding the moment about each of the three coordinate planes to find the x,y, and z component of the center of mass.

Center of Mass and Moments of Inertia of Solids in 3-dimensions

The density function of a solid is $\rho(x,y,z)$ in units of mass per unit volume

Mass= density*volume

$\iiint_E \rho(x,y,z)dV$ where E is the solid over which we integrate

Moments about the three coordinate planes are

$$M_{yz} = \iiint_E x\rho(x,y,z)dV$$

$$M_{xz} = \iiint_E y\rho(x,y,z)dV$$

$$M_{xy} = \iiint_E z\rho(x,y,z)dV$$

Center of mass $(\bar{x},\bar{y},\bar{z})$ of a solid is located at

$$\bar{x} = \frac{M_{yz}}{m} \qquad \bar{y} = \frac{M_{xz}}{m} \qquad \bar{z} = \frac{M_{xy}}{m}$$

**If we were to assume that the density of each of the models is constant throughout then we would integrate x,y,z separately over the bounds of the solid, pulling the density out as a constant.

For the conical frustum

We would use cylindrical coordinates to describe a cone along the z-axis and integrate using (r, θ ,z). However, because of symmetry along the z-axis to assume that the x and y coordinates of the centroid is 0 because the density is constant. By pulling the density constant out, the triple integral of dV is the volume, leaving us with density*volume to find the mass.

$$mass = [\tfrac{1}{3}\pi h(R_1^2 + R_1 R_2 + R_2^2)] * k$$

Let k be a constant density, and we use the area formula for a conical frustum found in the modeling section of this paper

We can model the z component using a single integral because we have already determined the x and y components of the centroid to be 0 due to symmetry

For simplification processes, we obtain

$$\overline{z} = \frac{1}{V} \int_0^h z\pi[R_1 + (\ R_2 - R_1)\tfrac{z}{h}\ )]^2 dz$$

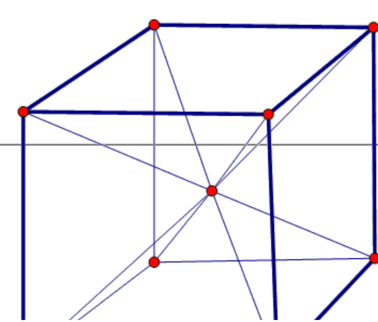
$$\overline{z} = \frac{h(3R_2^2+2R_2R_1+R_1^2)}{4(R_2^2+R_2R_1+R_1^2)}$$

To compare this to a cylinder, we can understand that the centroid of a cylinder is directly in the center of the horizontal plane that defines the cross-section that splits the volume in two. This is due to the symmetry of the solid and assumed constant density.

Because of this, we are able to establish that the center of gravity of a conical frustum is lower than the center of gravity of a cylinder, indicating that the frustum is more stable.

Deriving an equation for the centroid of a square pyramidal frustum is significantly more difficult to prove because its surface area is written in terms of the lengths of the base sides which are lines of components x and y. However, without deriving an equation, we can apply the idea previously discussed with the cone and conical frustum. With a square pyramidal frustum, the distribution of mass is bottom-heavy so that the center of gravity is lower than that of a cube in which we can understand that the centroid of a cube of constant density is the point of intersection of the diagonals that connect opposing points on parallel planes (as pictured in the diagram below).

The center of mass of a cube is centrally located.



https://www.quora.com/How-many-diagonals-are-there-in-a-cube

From understanding the center of gravity for various 3-dimensional shapes and how they compare, we were able to conclude that a bottom-heavy structure with slanted sides is preferable to enhance the stability of the sandcastle foundation.

Initially, we thought a triangular pyramidal frustum would be an ideal shape, as the triangle is known as one of the strongest shapes. The idea of a structure made of only triangles was promising. However, the benefit of a square base with a pyramidal frustum is that it creates a wider base for the castle to sit atop. There are four points and four sides that will receive the most stress. These points are susceptible to movement if force is applied, such as rain or tides. This weakness of the structure led us to choose model 2, the conical frustum.

When deciding between a pyramidal frustum and a conical one, we had to consider the strength of the cross-sectional shapes and the ability of sand grains to fit these shapes. We considered the strengths and weaknesses of using a square or circular cross-section. It is evident that when solely comparing a circle and a square, the circle is the stronger shape due to the force distribution along its curvature. Force or stress on a circle is uniformly distributed so that no one point experiences more force than another. When considering the stress distribution on a square, it is inherently uneven as force applied to a side creates a greater strain on the vertices of that side and force applied to a vertice can result into collapsing and dysmorphia. However, the square base does provide structure that can be deemed significant in the case of small granule materials. The strength of a square base would provide the sand a point of stress (at each intersection of sides when a sandcastle is built atop the foundation. In this case, if the structure began to fall, the sand would move towards these points of stress, and discourage collapsing.

Despite this, we concluded the conical frustum is the best 3-dimensional shape suited for a sandcastle foundation. The circular cross-sections ensure that there is equal distribution of force along the circumference and it provides the best equipped for the environmental conditions.

The factors we have focused on are rain, erosion by tides, and wind. To intuitively understand how these factors will affect our foundation, we can consider the force interactions between the foundation and the water. The force of rain is distributed fairly evenly across a small surface area such as our foundation. In this case, if a pyramidal shape was considered it would most likely become a conical frustum because the corners of the pyramid would come into contact with the corners first. A conical structure provides the best ability of moving water off of the structure as quickly as possible before it absorbs too much water.

Improvements:

In our first model of a pyramidal frustum with a square base, the shape seemed especially weak to external forces (the elements). We improved upon this by increasing the number vertical faces of the shape so that there would be overall, a reduced amount of contact with water, wind, etc. The conical frustum provides this reduced contact with the elements.

Other improvements would be to plan for the foreseeable weather forecast and construct your sandcastle and foundation accordingly. With rain or increased humidity, it would be important to have a dryer surface so that the additional water would be absorbed by either the castle or the fondation. If there is a warmer or dryer forecast in the foreseeable future, plan to have the surface of the sandcastle and the foundation be moist on the surface.

There are, of course, other improvements that could have been made to the process and procedure of creating and making the best model for a sandcastle's foundation. With more testing, experimentation and time comes a greater model for the perfect sandcastle foundation.

FUTURE WORK

In conducting research on the *best* 3-dimensional geometric shape to use for a sandcastle foundation, we found a number of ideas that would need more research:

1. Look at the oblique rectangular pyramid frustum and oblique conical circular frustum and the strength of a foundation with this shape.
2. Consider different base polygons for the frustum(s): an elliptic conical frustum, a trapezium pyramid frustum, etc.
3. Take a closer look at the density of the shape of the foundation, a major contributor to the overall strength of the shape.
4. Consider a foundation that is simply a mound of sand with no particular shape (as seen in the first picture in the attached article).

Future work would also include lengthening the paper, while at the same time, improving its strength, depth, and general efficiency when it comes to writing clear and concise explanations and expounding upon relevant information.

RELEVANT EQUATIONS:

Angle of Repose

$arctan(\theta) = \mu_s$

$\theta = arctan(\frac{h}{r})$

θ: angle of repose

h: height of a cone

r: radius of a conic base

μ: coefficient of static friction between granules of sand

ARTICLE

Research: The Perfect Sandcastle Foundation

By: Catherine Gibson, Patrick Walker, & Haoqiaomiao Zheng

No matter if we prefer playing beach volleyball or simply enjoying the sunshine, sand castles are always one of the unique views by the seaside. While you were having fun building the sand castles, have you ever thought of how to make the most professional and stable of sand castles? Luckily, through our research and experiments, we have captured an insight into the secret of making the longest lasting sandcastle. And the answer to creating a perfect sandcastle is having the strongest foundation–a conical frustum. What is a conical frustum? Take one of those cones you just bought from the ice cream stand, turn it upside down (after you've finished the ice cream of course) and cut through the cone parallel to the bottom. Congratulations! You've now created a conical frustum, a ROLOⓇ-like shape that is perfect for your sandcastles foundation.

The very first thing you may have thought of was not the shape of the foundation but the sand itself. What type of sand are you supposed to use and how much water should be added to the sand? In our research we looked at how sand sticks together and why it's so special when it comes to making sandcastles. We were looking to answer the question–what about our world lets us create sand formations (including our castles)? How is it possible that these sand formations are able to stand up and what is it about extremely tiny solids that allows us to form such structures? We were exploring the physics and formation of these sandcastles, a bunch of very interesting, yet admittedly boring concepts when all we want to do is play in the sand and make the strongest and longest lasting sandcastles (and castle foundations)!

Besides the importance of using a conical frustum as the shape of the base, sand-to-water proportion should also be taken into consideration. As we all know, adding a little water to the sand, just like what we usually would do when we are making sandcastles on the beach, would always make the sand more sticky to each other so that the castle would be more stable. One interesting result from our study is that, when you add roughly 8 sand to 1 water, your castle would be the strongest! Even if you had no idea how this proportion would approximately be like, it's totally fine. You could just simply add less water, since the air humidity by the seaside is fairly higher than inland and if it rains a little to increase the amount of water contained in your castle, that would be even better! And also, keep in mind that it is always good to keep reshaping your castle from time to time.

In our experiments, we assumed that all tests for different foundations would use the same type of sand, on the same beach, at the same spot, at the same time and appropriately, the same sand-to-water ratio. And yes, all our sandcastle foundations (and the sandcastles themselves) all faced the sea, because, who wouldn't want to have a view of the ocean from your castle?

Here's a short explanation as to why the conical frustum is the best shape for your sandcastle's foundation:

It is the most weather resistant, sturdy base that can support any proportionally sized sandcastle. The slanted base reduces overall exposure to the elements.

The best foundation for your sandcastle:

- SHAPE: conical frustum proportional to the size of your castle
- SAND-to-WATER: 8 parts sand to 1 part water
- WHERE: your local beach (or sandbox if you really wanted to!)
- WHEN: any time of day, during all tides and all weather conditions
- WHY: to have the longest lasting sandcastle and the strongest sandcastle foundation
- TIPS: build an additional moat and/or wall around your foundation/castle to repel water

Now you know the inside scoop on building the best sandcastle and sandcastle foundation!

Looking forward to having fun in the sun? Send a picture of your sandcastle to funinthesun@gmail.com and it may get featured in next month's edition! Here is last month's submission from Johnny in Ventura, CA:

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